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SOLVABILITY OF THE $\bar\partial$ PROBLEM WITH C^∞ REGULARITY UP TO THE BOUNDARY ON WEDGES OF $\mathbb{C}^N$

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ABSTRACT

For a *wedge* W of $\mathbb{C}^N$, we introduce an intrinsic condition of weak q-pseudoconvexity which can be expressed in terms of q-subharmonicity both of a defining function or an exhaustion function. Under this condition we prove solvability of the $\bar\partial$ system for forms with $C^\infty(\bar W)$-coefficients of degree $\geq q+1$. Our method relies on the L^2-estimates by Hörmander. For $C^\infty(W)$ solvability we refer to Hörmander (if $\partial W \in C^2$), and to Zampieri (for general wedges W). For $C^\infty(\bar W)$ solvability and with $\partial W \in C^2$, we refer to Dufresnoy (if $q = 0$), Michel (if the number of negative Levi-eigenvalues of ∂W is constant), and finally Zampieri (for more general q-pseudoconvexity).

1. q-pseudoconvexity of wedges

Let W_i, $i = 1, \dots, m$, be C^2 half-spaces in a neighborhood of a point z_o and let W be the "wedge" defined by $W = \bigcap_i W_i$. We denote by M_i the hypersurfaces $M_i = \partial W_i$, by $\hat M_i$ the "faces" $\hat M_i = M_i \cap \partial W_i$ and also set

$$R = \{z \in M_i \cap M_j \text{ for some } i \neq j\} \quad \text{and} \quad \hat R = R \cap \partial W.$$

We suppose that the M_i's intersect transversally and that $\bigcap_{i=1}^m M_i$ is "generic". We take equations $r_i = 0$ for the M_i's (with the W_i's defined by $r_i < 0$) and define the Levi form of the function r_i (resp. of the hypersurface M_i) by $\bar\partial\partial r_i$ (resp. $\bar\partial\partial r_i|_{\partial r_i^\perp}$). Here " $\perp$" denotes the complex orthogonal. We now formulate our main assumption. For an orthonormal system of $(1,0)$-forms $\{\omega'\} = \{\omega_1, \dots, \omega_q\}$ on ∂W in a neighborhood U of z_o, with $\mathrm{Span}\{\partial_{\omega'}\}|_{\hat M_i} \subset T^{(1,0)}M_i\ \forall i$, which is

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$C^0(\partial W) \cap C^2(\partial W \setminus R)$ with bounded first and second derivatives, we have on ∂W:

$$\text{(1.1)} \qquad \begin{array}{l} \mathrm{Span}\{\partial_{\omega'}\} \text{ is a } q\text{-space of minimal trace} \\ \text{for } \bar\partial\partial r_i|_{\partial r_i^\perp} \text{ and } \bar\partial\partial r_i|_{\{\omega', \partial r_i\}^\perp} \geq 0. \end{array}$$

Let $\mu_i^1 \leq \mu_i^2 \leq \cdots$ be the eigenvalues of $\bar\partial\partial r_i|_{\partial r_i^\perp}$. Then (1.1) means that $\mathrm{Span}\{\partial_{\omega'}\}$ is engendred by q eigenvectors with eigenvalues $\mu_1, \dots, \mu_q$ and that $\mu_{q+1} \geq 0$. Another condition equivalent to (1.1) is that for an orthonormal completion $\{\partial_{\omega''}\}$ of $\{\partial_{\omega'}\}$ on each complex bundle $T^{(1,0)}M_i|_{\hat M_i}$, we have in a neighborhood U of z_o:

$$\text{(1.2)} \qquad \begin{array}{r} \bar\partial'\partial'' r_i(z) = 0, \;\; \bar\partial'\partial' r_i(z)(\bar w', w') \leq \bar\partial''\partial'' r_i(z)(\bar w'', w''), \quad \bar\partial''\partial'' r_i(z) \geq 0 \\ \forall z \in \hat M_i \cap U \, \forall i, \; \forall w = (w', w'') \in \mathbb{C}^N, |w'| = 1, |w''| = 1. \end{array}$$

Remark that (1.1) and (1.2) do not depend on the choice of the equations $r_i = 0$ for the M_i's.

Example: Let ∂W be C^4. Then the space spanned by the q eigenvectors corresponding to $\mu^1(z) \leq \cdots \leq \mu^q(z)$ depends C^2 on z both for $q = N - 1 - s^+_{\partial W}(z_o)$ and $q = s^-_{\partial W}(z_o)$ (because in both cases $\mu^q(z) < \mu^{q+1}(z)$). As for the second condition $\mu^{q+1}(z) \geq 0$, this holds (even with strict inequality) for $q = N-1-s^+_{\partial W}(z_o)$ but not for $q = s^-_{\partial W}(z_o)$. However, it holds in this second case if we make the additional assumption $s^-_{\partial W}(z) \equiv \mathrm{const}(= q)$. This is the situation treated by Michel in [8].

Let us represent, in complex coordinates $z = x + \sqrt{-1}y \in \mathbb{C}^N$, the boundary ∂W as a graph $x_1 = h$ ($h = h(y_1, z_2, \bar z_2, \dots)$) and the domain W as $x_1 > h$ with $x_1 - h$ inducing the parametric representation $x_1 = h_i$ on each $\hat M_i$. We put $r := -x_1 + h$, $\delta := -r$, $\phi := -\log\delta + \lambda|z|^2$. Let $S := \{z\colon h_i = h_j \text{ for } i \neq j\}$; clearly $S = R + \mathbb{R}^+_{x_1}$ and $\delta \in C^2(W \setminus S)$. S is a manifold (because the M_i's intersect transversally), and at each regular point of S (where $h_i = h_j$ but $h_i \neq h_k \, \forall k \neq j$) we consider the conormal

$$n_S = \frac{\partial(h_i - h_j)}{|\partial(h_i - h_j)|}.$$

Denote by $J(\cdot)$ the jump between the i's and the j's side of S. We have

$$\text{(1.3)} \qquad n_S = \frac{J(\partial r)}{|J(\partial r)|} = \frac{J(\partial\phi)}{|J(\partial\phi)|}.$$

It is also clear that $\partial_{\omega'}|_{\hat M_i\cap\hat M_j}\subset T^{(1,0)}(M_i\cap M_j)$. Thus if we define $\omega'(z)$ on $\bar W$ by $\omega'(z)=\omega'(z^*)$ (where $z\mapsto z^*$ is the projection on ∂W along the x_1-axis), then we have

$$\partial_{\omega'}|_S\subset T^{(1,0)}S. \tag{1.4}$$

We denote by $\omega''(z)$ an orthonormal completion of $\omega'(z)$ by (1,0)-forms. For ordered indices $J=(j_1<\cdots<j_k)$ of length $|J|=k$, we consider vectors $w=(w_J)$ and, for a permutation σ, we put $w_{\sigma(J)}=\operatorname{segn}\sigma\, w_J$.

THEOREM 1.1: *Let (1.1) be fulfilled for $\{\omega'\}$ of rank q. Let $\phi=-\log\delta+\lambda|z|^2$ be defined as above, and let (ϕ_{ji}) be the matrix of $\bar\partial\partial\phi$ in the basis $\omega=(\omega',\omega'')$. Then for a suitable λ, for a suitable neighborhood U of z_o, and for any $k\geq q+1$:*

$$\sideset{}{'}\sum_{|K|=k-1}\sum_{ij=1,\dots,N}\phi_{ji}w_{iK}\bar w_{jK}-\sideset{}{'}\sum_{|J|=k}\sum_{j\leq q}\phi_{jj}|w_J|^2\geq\lambda|w|^2 \tag{1.5}$$
$$\forall z\in W\cap(U\setminus S)\,\forall w.$$

Proof: (Cf. also [12] for a similar argument.) We first observe that we have

$$\bar\partial\partial r(z)=\bar\partial\partial r(z^*),\quad \partial r^\perp(z)=\partial r^\perp(z^*)\quad\forall z\in W. \tag{1.6}$$

Hence (1.6) permits to *propagate* (1.1) from $\partial W\setminus R$ to $W\setminus S$. We remark that

$$\bar\partial\partial\phi(z)=\delta^{-1}\bar\partial\partial r+\delta^{-2}\bar\partial r\wedge\partial r+\lambda d\bar z\wedge dz. \tag{1.7}$$

Let $\lambda_1(z)\leq\lambda_2(z)\leq\cdots$ and $\mu_1(z)\leq\mu_2(z)\leq\cdots$ be the eigenvalues of $\bar\partial\partial\phi(z)$ and $\bar\partial\partial r(z)|_{\partial r^\perp(z)}$ respectively. It is clear that $\delta^{-1}\mu_i+\lambda$ are the eigenvalues of $\bar\partial\partial\phi|_{\partial r^\perp}$. We have

$$\sideset{}{'}\sum_{|K|=k-1}\sum_{ij=1,\dots,N}\phi_{ji}\bar w_{jK}w_{iK}\geq\Big(\sum_{j=1,\dots,k}\lambda_j\Big)|w|^2. \tag{1.8}$$

Also (1.1) gives, with the aid of (1.7),

$$\sideset{}{'}\sum_{|J|=k}\sum_{j\leq q}\phi_{jj}|w_J|^2=\Big(\delta^{-1}\sum_{j\leq q}\mu_j+q\lambda\Big)|w|^2.$$

Let ∂^τ (resp. ∂^ν) be the (1,0) derivatives tangential (resp. normal) to $\partial r^\perp(z)$. It is clear from (1.7) that for suitable c

$$\bar\partial\partial\phi\geq\delta^{-1}\bar\partial^\tau\partial^\tau r-c d\bar z^\tau\wedge dz^\tau+\lambda' d\bar z\wedge dz. \tag{1.9}$$

Let $\{N_k\}$ denote the k-dimensional planes of $\mathbb{C}^N$. We then have

$$(1.10)\qquad \begin{aligned}\sum_{j=1,\dots k}\lambda_j &= \inf_{N_k}\operatorname{trace}\left(\bar\partial\partial\phi|_{N_k}\right)\\ &\geq \inf_{N_k}\operatorname{trace}\left(\delta^{-1}\bar\partial^\tau\partial^\tau r - c\mathrm{d}\bar z^\tau\wedge\mathrm{d}z^\tau + \lambda\mathrm{d}\bar z\wedge\mathrm{d}z\right)\\ &\geq (k\lambda - kc) + \delta^{-1}\sum_{j=1,\dots,k}\mu_j.\end{aligned}$$

Thus for the new λ' defined by $\lambda' = (k-q)\lambda - kc$, we have

$$(1.11)\qquad \sum_{j=1,\dots k}\lambda_j - \delta^{-1}\sum_{j=1,\dots q}\mu_j - q\lambda \geq \lambda' + \delta^{-1}\sum_{j=q+1,\dots,k}\mu_j \geq \lambda',$$

where the last inequality follows from the second condition in (1.1). ∎

Here is our main result:

THEOREM 1.2: *Let W be a wedge of $\mathbb{C}^N$ defined by $r_j < 0$ $j = 1,\dots,m$ in a neighborhood of a point $z_o \in \partial W$, and assume (1.1) be fulfilled for a convenient orthonormal system of (1,0)-forms $\{\omega_j\} = \{\omega_1,\dots,\omega_q\}$ with $\partial_{\omega'}|_{\partial W\setminus R} \subset T^{(1,0)}(\partial W\setminus R)$ and whose coefficients belong to $C^0(\partial W)\cap C^2(\partial W\setminus R)$ and have bounded first and second derivatives. Then there exists a fundamental system of neighborhoods $\{U\}$ of z_o such that for any $\bar\partial$-closed form f of degree $k \geq q+1$ with $C^\infty(\overline{W\cap U})$ coefficients, there is a form u of degree $k-1$ with $C^\infty(\overline{W\cap U})$-coefficients, which solves the equation $\bar\partial u = f$.*

2. L^2 estimates and proof of Theorem 1.2

For a real positive function ϕ, we define $L^2_\phi(W)$ to be the space of functions on W which are square integrable in the measure $e^{-\phi}\mathrm{d}V$ ($\mathrm{d}V$ being the Euclidean element of volume). We denote by $||\cdot||_\phi$ the corresponding norm. We denote by $L^2_\phi(W)^k$ the space of $(0,k)$-forms with coefficients in $L^2_\phi(W)$. In a basis $\{\omega_j\}$ they are written as $f = \sum'_{|J|=k} f_J\bar\omega_J$ where $\sum'$ denotes summation over ordered indices and where $\bar\omega_J = \bar\omega_{j_1}\wedge\cdots\wedge\bar\omega_{j_k}$. We denote by (ϕ_{ji}) the matrix of the Hermitian form $\bar\partial\partial\phi$. (We assume that the basis $\{\omega\}$ is chosen as a completion of the system $\{\omega'\}$ in which (1.1) and hence (1.5) hold.) We introduce also another function $\psi \geq 0$, and consider the complex of closed densely defined operators

$$(2.1)\qquad L^2_{\phi-2\psi}(W)^{k-1} \overset{\bar\partial}{\to} L^2_{\phi-\psi}(W)^k \overset{\bar\partial}{\to} L^2_\phi(W)^{k+1}.$$

We denote by $\bar\partial^*$ the adjoint of $\bar\partial$ and define the operator $\delta_{\omega_j}(\cdot) = e^{\phi}\partial_{\omega_j}(e^{-\phi}\cdot)$. We have, for $f \in C_c^\infty(W)^k$,

$$
(2.2)\qquad
\begin{aligned}
\bar\partial^* f =& -\sum_{|K|=k-1}{}' \sum_{j=1,\dots,N} e^{-\psi}\delta_{\omega_j}(f_{jK})\bar\omega_K \\
& -\sum_{|K|=k-1}{}' \sum_{j=1,\dots,N} e^{-\psi} f_{jK}\partial_{\omega_j}\psi\bar\omega_K + e^{-\psi}R_f, \\
\bar\partial f =& \sum_{|J|=k}{}' \sum_{j=1,\dots,N} \bar\partial_{\omega_j}(f_J)\bar\omega_j\wedge\bar\omega_J + R_f,
\end{aligned}
$$

where R_f are errors which involve products of the f_J's by derivatives of coefficients of the ω_j's. On the other hand we have

$$
(2.3)\qquad
\begin{aligned}
&\Big\| \sum_{|K|=k-1}{}' \sum_{j=1,\dots,N} \delta_{\omega_j}(f_{jK})\bar\omega_K \Big\|_\phi^2 = \\
&\qquad \sum_{|K|=k-1}{}' \sum_{ij=1,\dots,N} \int_W e^{-\phi}\delta_{\omega_i}(f_{iK})\overline{\delta_{\omega_j}(f_{jK})}\mathrm{d}V, \\
&\Big\| \sum_{|J|=k}{}' \sum_{j=1,\dots,N} \bar\partial_{\omega_j}(f_J)\bar\omega_j\wedge\bar\omega_J \Big\|_\phi^2 = \\
&\qquad -\sum_{|K|=k-1}{}' \sum_{ij=1,\dots,N} \int_W e^{-\phi}\bar\partial_{\omega_j}(f_{iK})\overline{\bar\partial_{\omega_i}(f_{jK})}\mathrm{d}V \\
&\qquad +\sum_{|J|=k}{}' \sum_{j=1,\dots,N} \int_W e^{-\phi}|\bar\partial_{\omega_j}(f_J)|^2\mathrm{d}V.
\end{aligned}
$$

It follows that

$$
(2.4)\qquad
\begin{aligned}
&\sum_{|K|=k-1}{}' \sum_{ij=1,\dots,N} \int_W e^{-\phi}\big(\delta_{\omega_i}(f_{iK})\overline{\delta_{\omega_j}(f_{jK})} - \bar\partial_{\omega_j}(f_{iK})\overline{\bar\partial_{\omega_i}(f_{jK})}\big)\,\mathrm{d}V \\
&+\sum_{|J|=k}{}' \sum_{j=1,\dots,N} \int_W e^{-\phi}|\bar\partial_{\omega_j}(f_J)|^2\mathrm{d}V \\
&\le 3\|\bar\partial^* f\|_{\phi-2\psi}^2 + 2\|\bar\partial f\|_\phi^2 + \sigma_1^2\|f\|_\phi^2 + 3\||\partial\psi| f\|_\phi^2 \quad \forall f\in C_c^\infty(W)^k,
\end{aligned}
$$

where σ_1 can be estimated by the sup-norm of the derivatives of the coefficients of the ω_j's over the support of f. (We shall also use the notation σ_2 as for the second derivatives.) We have the commutation relations

$$
(2.5)\qquad
\begin{aligned}
[\delta_{\omega_i}, \bar\partial_{\omega_j}] &= \partial_{\bar\omega_j}\partial_{\omega_i}\phi + \sum_h c_{ji}^h\partial_{\omega_h} - \sum_h \bar c_{ij}^h\partial_{\bar\omega_h} \\
&= \phi_{ji} + \sum_h c_{ji}^h\delta_{\omega_h} - \sum_h \bar c_{ij}^h\partial_{\bar\omega_h},
\end{aligned}
$$

where the c^h_{ji}'s involve the antiholomorphic derivatives of the coefficients of the ω_i's. We apply (2.5) to the whole of the terms in the first sum on the left of (2.4) and to the terms with $j \leq q$ in the second. We obtain

$$\begin{aligned}(2.6)\quad &\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \cdot + \sum_{|J|=k}' \sum_{j=1,\dots,N} \cdot \\ &= \Big(\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_\Omega e^{-\phi} \phi_{ji} f_{iK} \bar{f}_{jK} \mathrm{dV} - \sum_{|J|=k}' \sum_{j\leq q} \int_\Omega e^{-\phi} \phi_{jj} |f_J|^2 \mathrm{d}V \Big) \\ &+ \Big(\sum_{|J|=k}' \sum_{j \leq q} \|\delta_{\omega_j} f_J\|^2_\phi + \sum_{|J|=k}' \sum_{j\geq q+1} \|\partial_{\bar\omega_j} f_J\|^2_\phi \Big) \\ &+ \Big(\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_S e^{-\phi} J(\partial_{\omega_i}\phi) \bar{n}_j f_{iK} \bar{f}_{jK} \mathrm{d}S \\ &- \sum_{|J|=k}' \sum_{j\leq q} \int_S e^{-\phi} J(\partial_{\omega_j}\phi) \bar{n}_j |f_J|^2 \mathrm{d}S \Big) + \text{Error},\end{aligned}$$

where the error term has the estimate

$$(2.7)\quad |\text{Error}| \leq \Big(\sum_{|J|=k}' \sum_{j\leq q} \|\delta_{\omega_j} f_J\|^2_\phi + \sum_{|J|=k}' \sum_{j\geq q+1} \|\bar\partial_{\omega_j} f_J\|^2_\phi \Big) + (\sigma_1^2 + \sigma_2) \|f\|^2_\phi.$$

Note that $n' = 0$ whence $\sum'_{|J|=kj\leq q} \sum \int_S \cdot \mathrm{d}S = 0$. Moreover, since $n = \frac{J(\partial\phi)}{|J(\partial\phi)|}$, then

$$\begin{aligned}(2.8)\quad \sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_S \cdot \mathrm{d}S &= \sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_S e^{-\phi} n_i \bar{n}_j |J(\bar\partial\phi)| \bar{f}_{jK} f_{iK} \mathrm{d}S \\ &= \sum_{|K|=k-1}' \int_S e^{-\phi} \Big| \sum_{i=1,\dots,N} n_i f_{iK} \Big|^2 |J(\bar\partial\phi)| \mathrm{d}S \geq 0.\end{aligned}$$

We did not use any property of ϕ so far. For the sequel, however, we need our main hypothesis. We assume that (1.5) is fulfilled for any $z \in W \setminus S$, apply for $w = (f_J)_J$, and get

(2.9)

$$\sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \int_W e^{-\phi} \phi_{ji} f_{iK} \bar{f}_{jK} \mathrm{dV} - \sum_{|J|=k}' \sum_{j\leq q} \int_W e^{-\phi} \phi_{jj} |f_J|^2 \mathrm{d}V \geq \lambda \|f\|^2_\phi.$$

By combining (2.7), (2.8), (2.9), we obtain the following estimation for the first line of (2.6):

$$(2.10)\qquad \sum_{|K|=k-1}' \sum_{ij=1,\dots,N} \cdot + \sum_{|J|=k}' \sum_{j=1,\dots,N} \cdot \geq \lambda \|f\|^2_\phi - (\sigma_1^2 + \sigma_2) \|f\|^2_\phi.$$

By plugging together (2.4) and (2.10) we get

(2.11)

$$\lambda||f||^2_\phi \le 3||\bar\partial^* f||^2_{\phi-2\psi} + 2||\bar\partial f||^2_\phi + (\sigma_1^2+\sigma_2)||f||^2_\phi + 3|||\partial\psi|f||^2_\phi \quad \forall f \in C_c^\infty(W)^k.$$

We fix now a compact subset $K \subset\subset W$ that we may assume in the form $K = \{\phi \le n\}$ for some n, and choose ψ according to [6, Lemma 4.1.3] (in particular we can choose $\psi|_K \equiv 0$). This ensures density of C_c^∞ into L^2-forms. Thus now (2.11) holds for L^2 instead of C_c^∞ forms. We replace the above ϕ by

$$\chi(\phi) + (3+\sigma_1^2+\sigma_2)|z|^2,$$

where χ is a positive convex function of a real argument t which satisfies:

$$\text{(2.12)} \qquad \begin{cases} \chi(t) \equiv 0, & \text{for } t \le n, \\ \dot\chi(t) \ge \sup\limits_{\{z:\phi(z)\le t\}} \frac{3(|\partial\psi|^2+e^\psi-1)}{\lambda}, & \text{for } t \ge n. \end{cases}$$

Under this choice of ϕ and ψ we conclude, for $k \ge q+1$,

$$\text{(2.13)} \qquad ||f||^2_{\phi-\psi} \le ||\bar\partial^* f||^2_{\phi-2\psi} + ||\bar\partial f||^2_\phi \quad \forall f \in D^k_{\bar\partial} \cap D^k_{\bar\partial^*},$$

where $D^k_{\bar\partial}$ and $D^k_{\bar\partial^*}$ are the domains in $L^2_\phi(W)^k$ of $\bar\partial$ and $\bar\partial^*$, respectively. Moreover, for any compact subset $K \subset\subset \Omega$, we may choose $\psi|_K \equiv 0$ and $\phi|_K \equiv (3+\sigma_1^2+\sigma_2)|z|^2$. To conclude the proof of Theorem 1.2, we need some further preparation which consists in a rearrangement of (2.13).

PROPOSITION 2.1: *Let W be bounded and endowed with an exhaustion function satisfying (1.5) for any $z \in W \setminus S$. Then for $k \ge q+1$, for a suitable constant c, and for any form $f \in L^2_{c|z|^2}(W)^k$ with $\bar\partial f = 0$, there exists $u \in L^2_{c|z|^2}(W)^{k-1}$ such that*

$$\text{(2.14)} \qquad (\bar\partial u = f\,,\ \bar\partial^* u = 0) \quad ||u||^2_{c|z|^2} \le ||f||^2_{c|z|^2}.$$

Cf. Appendix for the Proof.

Let $||\cdot||_{(s)}$ denote the norm in the Sobolev space $H^s(W)$. Let $W^\epsilon := \{z \in W : \mathrm{dist}(z, \partial W) > \epsilon\}$.

PROPOSITION 2.2: *Let W be bounded and endowed with an exhaustion function satisfying (1.5). Then for $k \ge q+1$ and for any $f \in C^\infty(W)^k$ with $\bar\partial f = 0$, there is $u \in C^\infty(W^\epsilon)^{k-1}$ such that for any s and for suitable M_s we have*

$$\text{(2.15)} \qquad (\bar\partial u = f\,,\ \bar\partial^* u = 0), \quad ||u||_{(s+1)} \le \frac{M_s}{\epsilon^{s+1}}||f||_{(s)}$$

(the norms of u and f being evaluated over W^ϵ and W respectively).

Cf. Appendix for the Proof.

End of Proof of Theorem 1.2: We suppose that W is locally defined at $z_0 = 0$ by $-x_1 + h < 0$ and then define W_ν by $-x_1 + h < \eta^{2^\nu}/2$ for $0 < \eta < \frac{1}{2}$. Clearly we have on a neighborhood of z_o

$$(2.16) \quad \{z \in \mathbb{C}^N\colon \mathrm{dist}(z, W) < \eta^{2^{\nu+1}}\} \subset W_\nu \subset \{z \in \mathbb{C}^N\colon \mathrm{dist}(z, W) < \eta^{2^\nu}/2\}.$$

We consider the sequence of domains $W_\nu \cap U_\nu \supset\supset W_{\nu+1} \cap U_{\nu+1} \supset\supset \cdots \supset\supset W \cap U$ where U_ν (resp. U) is the sphere with center z_o and radius $\rho + \eta^{2^\nu}/2$ (resp. ρ) for small ρ. We note that (1.5) is fulfilled by the exhaustion function $\phi := -\log(\delta + \eta^{2^\nu}/2) + \lambda|z|^2 + \log(-|z - z_o|^2 + (\rho + \eta^{2^\nu}/2)^2)$. Thus we can apply (2.15) with the pair W, W^ϵ replaced by $W_\nu \cap U_\nu$, $W_{\nu+1} \cap U_{\nu+1}$. It is then easy (cf. Appendix) to find a sequence of approximate solutions on the $W_\nu \cap U_\nu$ which converge in $\overline{W \cap U}$ to a true solution of $\bar\partial u = f$. ∎

3. Appendix

For the convenience of the reader we collect here the proofs of Proposition 2.1 ([6, Prop. 4.4.1], Prop. 2.2 ([6, Prop. 4.2.4]) and the argument of Dufresnoy [3].

Proof of Proposition 2.1: Let ϕ be an exhaustion function which satisfies (1.5) on the whole W and set $K_n := \{\phi \le n\}$ for n large. Take $\psi = \psi_n$ with $\psi|_{K_n} \equiv 0$ and $\chi = \chi_n$ convex with $\chi(t) \ge 0\ \forall t \in \mathbb{R}$, $\chi(t) \equiv 0\, \forall t \le n$ and which verifies (2.12). We also suppose $\chi(\phi) - 2\psi \ge 0$ and consider $\chi(\phi) + c|z|^2$ where c is an uniform upper bound for $2 + \sigma_1^2 + \sigma_2$ on the whole $\bar W$. Note that $\chi(\phi) + c|z|^2 \equiv c|z|^2$ over K_n.

Let $\int_W e^{-c|z|^2}|f|^2 \mathrm{d}V \le 1$ (hence $f \in L^2_{\chi(\phi)+c|z|^2-\psi}(W)^k$). For any $g \in L^2_{\chi(\phi)+c|z|^2-\psi}(W)^k$ we have

$$(3.1) \qquad \begin{aligned} |(f,g)_{\chi(\phi)+c|z|^2-\psi}|^2 &\le \int_W e^{-2\chi(\phi)-c|z|^2+2\psi}|g|^2 \mathrm{d}V \\ &\le \int_W e^{-\chi(\phi)-c|z|^2}|g|^2 \mathrm{d}V \\ &\le \|\bar\partial^* g\|^2_{\chi(\phi)+c|z|^2-2\psi} + \|\bar\partial g\|^2_{\chi(\phi)+c|z|^2}, \end{aligned}$$

where the last inequality follows from (2.13). It is immediate to see, because of $\bar\partial f = 0$, that in fact

$$(f, g)_{\chi(\phi)+c|z|^2-\psi} \le \|\bar\partial^* g\|_{\chi(\phi)+c|z|^2-2\psi}.$$

It follows that the mapping $\bar\partial^* g \mapsto (f,g)_{\chi(\phi)+c|z|^2-\psi}$ is represented by $\bar\partial^* g \mapsto (u, \bar\partial^* g)_{\chi(\phi)+c|z|^2-2\psi}$, for a convenient u which can be chosen in the image of $\bar\partial^*$ thus verifying $\bar\partial^* u = 0$ and with $||u||_{\chi(\phi)+c|z|^2-2\psi} \leq 1$. By approximating W by the sequence K_n, taking a sequence of solutions $u = u_n$, and remarking that $\chi_n(\phi) \equiv 0$ and $\psi_n \equiv 0$ on K_n, we can find a subsequence u_{n_j} which converges to a form u which fulfills all requirements in the statement. ∎

Proof of Proposition 2.2 (Dufresnoy [3]): Let $\chi^\epsilon \equiv 1$ on $\bar W^\epsilon$, $\operatorname{supp}\chi_\epsilon \subset W^{\epsilon/2}$. It is easy to find such a smooth χ^ϵ with the property

$$(3.2) \qquad |\bar\partial^\alpha \chi^\epsilon| \leq \frac{M_\alpha}{\epsilon^{|\alpha|}} \quad \forall \alpha \in \mathbb{N}^N.$$

Here we have used the notation $\bar\partial^\alpha = \bar\partial^{\alpha_1}_{\omega_1} \cdot \ldots \cdot \bar\partial^{\alpha_N}_{\omega_N}$. If we apply (2.4) to the (compactly supported) form $\bar\partial^\alpha(\chi^\epsilon u)$ and with $\phi = 0$, $\psi = 0$, we get

$$(3.3) \qquad ||\bar\partial^\alpha(\chi^\epsilon u)||^2_{(1)} \leq c(||\bar\partial^\alpha(\chi^\epsilon u)||^2_{(0)} + ||\bar\partial(\bar\partial^\alpha \chi^\epsilon u)||^2_{(0)} + ||\bar\partial^*(\bar\partial^\alpha \chi^\epsilon u)||^2_{(0)})$$

(where $||\cdot||_{(0)}$ denotes the norm in $H^0 = L^2$ and $\bar\partial^\kappa\colon (L^2)^\kappa \to (L^2)^{\kappa-1}$). In particular, suppose that u is a solution of $(\bar\partial u = f, \bar\partial^* u = 0)$ satisfying $||u||_{c|z|^2} \leq ||f||_{c|z|^2}$ according to Proposition 2.1. (Here $\bar\partial^\kappa\colon {L^2_{c|z|^2}}^\kappa \to {L^2_{c|z|^2}}^{\kappa-1}$.) By combining (3.2) with (3.3) for this particular choice of u, and observing that W being bounded, the $L^2_{c|z|^2}(W)$ and $L^2(W)$-norms are equivalent, we get at once the conclusion. ∎

Outline of the approximation argument in the end of the proof of Theorem 1.2: Let $W_\nu \supset\supset W_{\nu+1} \supset\supset \cdots \supset\supset W$ be a sequence of domains satisfying (2.16). Take a sequence of spheres $U_v \supset\supset U_{v+1} \supset\supset \cdots$ with center z_0 and radius $\rho + \eta^{2^\nu}/2$ and write again W_v instead of $W_v \cap U_v$. Let $f \in C^\infty(\bar W)^k$ satisfy $\bar\partial f = 0$. Extend f to $\tilde f$ such that

$$||\bar\partial \tilde f||_{(s)} \leq M_{rs}\eta^{r2^\nu} \text{ on } W_\nu \text{ for any } r,\ s \text{ and for suitable } M_{rs}.$$

This is clearly possible because $\bar\partial \tilde f \equiv 0$ on W and because

$$W_\nu \subset \{z\colon \operatorname{dist}(z, W) < \eta^{2^\nu}/2\}.$$

According to Proposition 2.2, there is a solution h_ν on $W_{\nu+1}$ of

$$\begin{cases} \bar\partial h_\nu = \bar\partial \tilde f \\ ||h_\nu||_{(s+1)} \leq M_s(\eta^{2^{\nu+1}})^{-s-1}||\bar\partial \tilde f||_{(s)} \end{cases}$$

(due to $W_{\nu+1} \subset \{z : \text{dist}(z, \partial W_\nu) > \eta^{2^{\nu+1}}/2\}$). Solve on W_2 the equation $\bar{\partial} u_1 = \tilde{f} - h_1$, and, inductively on $W_{\nu+2}$,

$$\bar{\partial} u_{\nu+1} = h_\nu - h_{\nu+1},$$

with the estimates

$$\begin{aligned} ||u_{\nu+1}||_{(s+2)} &\leq M_{s+1}(\eta^{2^{\nu+2}})^{-(s+2)}||h_\nu - h_{\nu+1}||_{(s+1)} \\ &\leq M'_s(\eta^{2^{\nu+2}})^{-2s-3} M_{rs}\eta^{r2^\nu} \\ &\leq M'_{rs}\frac{1}{2^\nu} \quad (r,\, \nu \text{ large}). \end{aligned}$$

Therefore $\sum_{\nu=1}^{\infty} u_\nu$ converges in $C^\infty(\bar{W})$ and solves on $\bar{W}$:

$$\bar{\partial}(\sum_{\nu=1}^{\infty} u_\nu) = \tilde{f} - \lim_\nu h_\nu = \tilde{f}.$$

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